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OBSERVATIONS ON $x^2 = y^{2a+1} + z^{2b+1}$

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Abstract
This paper aims at determining the non-zero distinct integer solutions to the Fermat equation of the form $x^2 = y^{2a+1} + z^{2b+1}$. In particular, we have presented integer solutions to two different choices of Fermat equations represented by $x^2 = y^{8b+5} + z^{2b+1}$ & $x^2 = y^{4A+1} + z^{4A+3}$. A few interesting relations between the solutions are also given.

Keyword: *Generalized Fermat equation, Diophantine equation, Integer solutions.*

I. INTRODUCTION

Let $p, q, r \in Z_{\geq 2}$. The ternary equation with exponents p, q, r represented by $x^p + y^q = z^r$ is known as the generalized Fermat equation which has been analyzed for many exponents (p,q,r) including varieties of infinite families of exponents {L.J.Mordell [4], Michael A.Bennett et al., [8], A.Kraus [15], B.Poonen et al.,[17], S.Abdelalim,H.Dyani [18]}. In [1], D.Brown has obtained Primitive integral solutions to $x^2 + y^3 = z^{10}$. In [2], N.Bruin has solved the equation $x^3 + y^9 = z^2$ for the primitive solutions. A complete solution to $X^2 + Y^3 + Z^5 = 0$ is solved by J.Edwards [3]. In [5,6], The generalized Fermat equations $x^3 + y^4 + z^5 = 0, X^2 + Y^3 = Z^{15}$ are analyzed by S.Siksek and M.Stoll. In [7,9], M.Bennett et al., solved the Diophantine equation $x^{2n} + y^{2n} = z^5$, $A^4 + 2^{\delta} B^2 = C^n$. In [10], N.Billerey, L.V.Dieulefait has analyzed Fermat-type equation $x^5 + y^5 = dz^p$. In [11], N.Bruin has solved the Diophantine equations $x^2 \pm y^4 = \pm z^6$ and $x^2 + y^8 = Z^3$ for the integer solutions. In [12,14], I.Chen, and H.Darmaon have considered the equation $a^2 + b^{2p} = c^5$, $x^n + y^n = z^2$ and $x^n + y^n = z^3$ for their integer solutions. In [13], S.Dahmen. solved the Diophantine equation $x^2 + y^{2n} = z^3$ for its integer solutions. In [16], B.Poonen has considered the Diophantine equations of the form $x^n + y^n = z^m$ for their integral solutions. A search is made for obtaining integer solutions to three special generalized Fermat equations $x^3 + y^4 = z^{13}, x^2 + y^3 = z^{15}$ and $x^4 = y^5 + z^7$ by M.A.Gopalan et al.,[20]. In [21], B.L.Bhatia and Supriya Mohanty have considered Nasty Numbers and their characterizations. Also, one may refer the website [19].

These results motivated us to search for integer solutions to the Fermat equation represented by $x^2 = y^{2a+1} + z^{2b+1}$. In this paper, a method has been described to solve this equation in integers for the choices of $a \& b$ given by $a = (2b+1)2^k$ & $a = 2A, b = 2A+1$. It seems that this set of solutions is not presented earlier. A few interesting relations between the solutions are also given.

II. METHOD OF ANALYSIS

The generalized Fermat equation to be solved is given by

$$x^2 = y^{2a+1} + z^{2b+1} \tag{1}$$

Assuming $x = n^{\alpha}(n+1)^{2ab+a+b+1}; y = n^{\beta}(n+1)^{2b+1}; z = n^{\gamma}(n+1)^{2a+1}$

in (1), we get $n^{2\alpha+1} + n^{2\alpha} = n^{(2a+1)\beta} + n^{(2b+1)\gamma}$

Equating the exponents on both sides, we have

$$(2\alpha+1) = (2a+1)\beta \tag{2}$$

$$(2\alpha) = (2b+1)\gamma \tag{3}$$

We present below two different methods of solving (2) and (3) leading to two different sets of solutions to (1).

Method 1:

The substitution $\gamma = 2^{k+1}, k \ge 0$

in (3) gives $\alpha = (2b+1)2^k$ and from (2), we have $\beta = \left(\dfrac{(2b+1)2^{k+1}+1}{(2a+1)}\right)$

In particular, when $a = (2b+1)2^k$, it is noted that $\alpha = (2b+1)2^k; \beta = 1; \gamma = 2^{k+1}$

Result: The values of x, y, z satisfying the considered equation are given by

$$x = n^{(2b+1)2^k}(n+1)^{2(2b+1)2^k b+(2b+1)2^k+b+1}$$

$$y = n(n+1)^{2b+1}$$

$$z = n^{2^{k+1}}(n+1)^{2(2b+1)2^k+1}$$

Illustration:

For $k = 1$ in the above solutions, we get the solutions of the equation $x^2 = y^{8b+5} + z^{2b+1}$ to be

$$x = n^{(4b+2)}(n+1)^{(8b^2+9b+3)} \tag{1a}$$

$$y = n(n+1)^{(2b+1)} \tag{1b}$$

$$z = n^4(n+1)^{(8b+5)} \tag{1c}$$

A few numerical examples are exhibited in the table below:

n	b	x	y	z
2	1	$2^6 * 3^{20}$	$2^1 * 3^3$	$2^4 * 3^{13}$
2	4	$2^{18} * 3^{167}$	$2^1 * 3^9$	$2^4 * 3^{37}$
3	2	$3^{10} * 4^{53}$	$3^1 * 4^5$	$3^4 * 4^{21}$
3	3	$3^{15} * 4^{102}$	$3^1 * 4^7$	$3^4 * 4^{29}$

Properties:

A few interesting properties among the solutions of (1a),(1b) and (1c) are presented below:

1. $n^3(n+1)^{6b+4}y = z$

   Proof: Dividing (1c) by (1b), the above result is obtained.

2. $xy^{4(8b^2+13b+5)} = (z-y^4)^{4b+2} z^{8b^2+9b+3}$

   Proof: $\dfrac{z}{y^4} = (n+1)$ (4)

   Therefore, $n = \dfrac{z-y^4}{y^4}$ (5)

   Using (4) and (5) in (1a), we get after simplification $xy^{4(8b^2+13b+5)} = (z-y^4)^{4b+2} z^{8b^2+9b+3}$

3. $xy^2 = z^{b+1}$

   Proof: $y^2 = n^2(n+1)^{4b+2}$

   Raising (1c) to the power b, we get $z^b = n^{4b}(n+1)^{8b^2+5b}$

   Therefore, $\dfrac{x}{y^2 z^b} = n+1 = \dfrac{z}{y^4}$ {by (4)}

   Thus, $xy^2 = z^{b+1}$

4. $y^{16b+18} = z^{4b+2}(z-y^4)^2$

   Proof: $y^2 = n^2(n+1)^{4b+2}$

   Raising (1b) to the power $4b+2$, we have

   $y^{4b+2} = n^{4b+2}(n+1)^{8b^2+8b+2}$

   Using (4) and (5), we have $y^{16b+18} = z^{4b+2}(z-y^4)^2$

5. $y^{16b+16} = xz^{3b+1}(z-y^4)^2$

   Proof: Raising (1c) to the power $b+1$, we get

   $z^{b+1} = n^{4b+4}(n+1)^{8b^2+13b+5} = xn^2(n+1)^{4b+2}$

   Employing (4) and (5), we have $y^{16b+16} = xz^{3b+1}(z-y^4)^2$

6. Each of the following represents a Nasty Number [21]

   i. $6\left(\dfrac{xy^{4(8b^2+13b+5)}}{z^{8b^2+9b+3}}\right)$

   ii. $6\left(y^{16b+18}\right)$

Proof for i:

In property 2, it is observed that $\left(\dfrac{xy^{4(8b^2+13b+5)}}{z^{8b^2+9b+3}}\right)$ is a perfect square.

Thus, $6\left(\dfrac{xy^{4(8b^2+13b+5)}}{z^{8b^2+9b+3}}\right)$ is a Nasty Number.

Proof for ii:

From property 5, it is seen that $\left(y^{16b+18}\right)$ is a perfect square. And thus, $6\left(y^{16b+18}\right)$ is a Nasty Number.

7. Each of the following represents a Perfect square
   i. $\left(\dfrac{y^{16b+18}}{z^{4b+2}}\right)$
   ii. $\left(\dfrac{y^{16b+16}}{xz^{3b+1}}\right)$

Proof for i: The above properties are readily obtained from properties 4 and 5.

8. $\left(y^{24b+27}\right)$ is a cubical integer

Proof for ii: Dividing (1c) by (1b), we have $\dfrac{z}{y} = n^3(n+1)^{6b+4}$

Using (4) and (5), it is noted that $y^{24b+27} = [(z-y^4)z^{2b+1}]^3$, which is a cubical integer.

Method 2:

Eliminating α between (2) & (3), we get

$$(2a+1)\beta-(2b+1)\gamma=1 \tag{5}$$

This is solved for suitable choices of a & b. In particular, taking $a=2A, b=2A+1$ in (5), it is satisfied by $\beta=2A+1, \gamma=2A$ and from (3), we have $\alpha=A(4A+3)$

Result: The solutions of $x^2 = y^{4A+1}+z^{4A+3}$ are given by

$$x=n^{(4A^2+3A)}(n+1)^{(8A^2+8A+3)} \tag{2a}$$

$$y=n^{(2A+1)}(n+1)^{(4A+3)} \tag{2b}$$

$$z=n^{2A}(n+1)^{(4A+1)} \tag{2c}$$

A few numerical examples are exhibited in the table below:

n	A	x	y	z
2	2	$2^{22}*3^{50}$	2^5*3^{11}	2^4*3^9
3	1	3^7*4^{18}	3^3*4^7	3^2*4^5
4	2	$4^{22}*5^{50}$	4^5*5^{11}	4^4*5^9
5	3	$5^{45}*6^{98}$	5^7*6^{15}	5^6*6^{13}

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Properties:

A few interesting properties among the solutions of (2a),(2b) and (2c) are presented below:

i. $\left(\dfrac{y}{nz}\right)$ is a perfect square

Proof: Dividing (2b) by n(2c), we get $\left(\dfrac{y}{nz}\right)=(n+1)^2$, a perfect square

ii. $6\left(\dfrac{y^{20A+9}}{z^{20A+9}n}\right)$ is a Nasty Number.

Proof: Dividing (2b) by (2c) and raising to the power $(20A+9)$, we get

$$\left(\frac{y^{20A+9}}{z^{20A+9}}\right)=n^{20A+9}(n+1)^{40A+18}$$

Therefore, $6\left(\dfrac{y^{20A+9}}{z^{20A+9}n}\right)$ is a Nasty Number.

iii. $\left(\dfrac{yz}{n^{4A+1}}\right)$ is a biquadratic integer.

Proof: Multiplying (2b) and (2c), we have $yz=n^{4A+1}(n+1)^{8A+4}$

Therefore, $\left(\dfrac{yz}{n^{4A+1}}\right)=[(n+1)^{2A+1}]^4$, a biquadratic integer

iv. $\left(\dfrac{n^A x}{(n+1)^{8A^2}z^2}\right)$ is a perfect square

Proof: $\left(\dfrac{x}{z^2}\right)=n^{4A^2-A}(n+1)^{8A^2+1}$

$$\left(\frac{n^A x}{(n+1)^{8A^2+1}z^2}\right)=\left(n^{2A^2}\right)^2$$, a perfect square

v. $\left(\dfrac{yz}{n^{4A+1}(n+1)^{2A+1}}\right)$ is a cubical integer.

Proof: (6) is written as

$$\left(\frac{yz}{n^{4A+1}(n+1)^{2A+1}}\right)=(n+1)^{6A+3}=\left[(n+1)^{2A+1}\right]^3$$, a cubical integer.

III. CONCLUSION

In this paper, we have considered the generalized Fermat equation $x^2 = y^{2a+1} + z^{2b+1}$ and obtained integer solutions for the choices of $a \& b$ given by $a = (2b+1)2^k$ & $a = 2A, b = 2A+1$. To conclude, one may attempt to find integer solutions to the considered Fermat equation for the other choices of $a \& b$.

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